

A TOY MODEL OF HYPERBOLOIDAL APPROACH TO QUASINORMAL MODES

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We consider a scalar field propagating in the static region of the two-dimensional de Sitter space. This simple system is used to illustrate the advantages of hyperboloidal foliations in the analysis of quasinormal modes.

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1. Introduction

Many physical systems respond to perturbations by oscillating at certain characteristic frequencies. For closed systems (such as a guitar string), these frequencies, called normal frequencies, are real and correspond to the eigenvalues of a self-adjoint operator. For open systems (such as waves scattering off an obstacle or a black hole), the characteristic frequencies, called quasinormal (or scattering) frequencies, are complex. An imaginary part of the quasinormal frequency determines the exponential decay of the amplitude of oscillation, which is due to the loss of energy by radiation. In the literature, quasinormal modes are traditionally defined by imposing an outgoing wave condition at infinity [1] (and, in the case of black holes, also an ingoing boundary condition at the horizon [2]), which implements the physical condition that nothing is ‘coming in from infinity’ (or from the horizon). While in most situations this definition works fine, it is not quite satisfactory both from the mathematical and physical viewpoints (for an excellent discussion of this issue, see [3]). The problem is that the unitary evolution, based on the standard constant time foliations of spacetime, does not provide a natural setting for understanding the dissipation-by-dispersion phenomena. This

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drawback can be remedied by using hyperboloidal foliations and the associated non-unitary evolution which inherently incorporates the loss of energy by radiation. In this formulation, the quasinormal modes can be defined as genuine eigenmodes of a certain non-self-adjoint linear operator. To our knowledge, the hyperboloidal approach to quasinormal modes was first suggested by Schmidt [4]. In the past decade, this idea has been implemented numerically [5, 6] and developed rigorously in the mathematical literature [3, 7, 8]; it also featured in the recent proof of non-linear stability of the Kerr–de Sitter black holes by Hintz and Vasy [9].

The purpose of this pedagogical note, addressed to physicists, is to present a very simple toy model illustrating the advantages of the hyperboloidal approach to quasinormal modes.

2. Setup

Consider a two-dimensional manifold $\mathcal{M} = \{t \in \mathbb{R}, x \in (-1, 1)\}$ with the metric

$$g = - (1 - x^2) dt^2 + (1 - x^2)^{-1} dx^2. \quad (1)$$

It corresponds to the static region of the two-dimensional de Sitter spacetime with constant scalar curvature $R(g) = 2$. We introduce the hyperboloidal foliation

$$\tau = t + \frac{1}{2} \log (1 - x^2) .$$

In terms of the coordinates (τ, x) , metric (1) takes the form of

$$g = - (1 - x^2) d\tau^2 - 2x d\tau dx + dx^2, \quad (2)$$

which is regular at the cosmological horizons $x = \pm 1$.

We are interested in the propagation of a scalar field with mass m on \mathcal{M} , as described by the Klein–Gordon equation

$$(\square_g - m^2) \phi = 0, \quad (3)$$

where $\square_g \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ and ∇_α denotes the covariant derivative with respect to the metric g . In terms of the coordinates (τ, x) , we have

$$-\partial_{\tau\tau}\phi - 2x\partial_{\tau x}\phi - \partial_\tau\phi + \partial_x \left[(1 - x^2) \partial_x \phi \right] - m^2\phi = 0, \quad (4)$$

$$\phi(0, x) = f(x), \quad \phi_\tau(0, x) = g(x). \quad (5)$$

We assume that the functions $f(x)$ and $g(x)$ are smooth on $x \in [-1, 1]$. Note that the curves $x = \pm 1$ are null; consequently, no boundary conditions are imposed at the endpoints $x = \pm 1$.

Multiplying equation (4) by $\partial_\tau \phi$, one gets the conservation law

$$\partial_\tau \rho + \partial_x j = 0, \quad (6)$$

where

$$\rho = \frac{1}{2} \left[(\partial_\tau \phi)^2 + (1 - x^2) (\partial_x \phi)^2 + m^2 \phi^2 \right],$$

and

$$j = x(\partial_\tau \phi)^2 - (1 - x^2) \partial_\tau \phi \partial_x \phi.$$

Integrating the conservation law (6) over the curve $\tau = \text{const.}$ and defining the Bondi energy $\mathcal{E} = \int_{-1}^1 \rho dx$, one gets

$$\frac{d\mathcal{E}}{d\tau} = -[\partial_\tau \phi(\tau, 1)]^2 - [\partial_\tau \phi(\tau, -1)]^2,$$

which shows that the Bondi energy \mathcal{E} decreases due the fluxes of outgoing radiation across the cosmological horizons. This indicates that for $\tau \rightarrow \infty$, the solution tends to a static equilibrium, which in the case of (4) is just a constant (equal to zero if m is non-zero).

In the following, we first discuss the case of $m = 0$, which can be solved explicitly, and then analyze the case of $m \neq 0$ using the Galerkin method.

3. Massless scalar field

In this section, we set $m = 0$ in equation (4). Then, in terms of double null coordinates (see Fig. 1)

$$U = (1 - x)e^{-\tau}, \quad V = (1 + x)e^{-\tau}, \quad (7)$$

equation (4) is equivalent to $\partial_{UV} \phi = 0$. Hence, the general solution can be written as

$$\phi(\tau, x) = \phi_1(U) + \phi_2(V), \quad (8)$$

where ϕ_1 and ϕ_2 are arbitrary functions. As a consequence, the solution of equation (4) with initial data (5) can be represented by the d'Alembert formula, which in the case at hand takes the form of

$$\phi(\tau, x) = \left(1 - \frac{1}{2}U\right) f(1 - U) + \left(1 - \frac{1}{2}V\right) f(V - 1) + \frac{1}{2} \int_{V-1}^{1-U} [g(z) - f(z)] dz.$$

It follows immediately from the above formula that the end-state of the evolution is a constant given by

$$\phi_\infty \equiv \lim_{\tau \rightarrow \infty} \phi(\tau, x) = f(1) + f(-1) + \frac{1}{2} \int_{-1}^1 [g(z) - f(z)] dz. \quad (9)$$

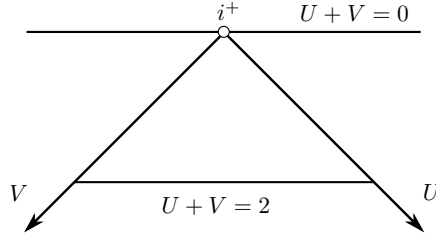


Fig. 1. Conformal diagram for metric (1).

In order to see the rate of convergence to the end-state, let us expand (8) in the Taylor series around $e^{-\tau} = 0$. We get

$$\begin{aligned}\phi(\tau, x) &= \phi_1 [(1-x)e^{-\tau}] + \phi_2 [(1+x)e^{-\tau}] \\ &= \phi_1(0) + \phi_2(0) + \phi_1'(0)(1-x)e^{-\tau} + \phi_2'(0)(1+x)e^{-\tau} \\ &\quad + \frac{1}{2}\phi_1''(0)(1-x)^2e^{-2\tau} + \frac{1}{2}\phi_2''(0)(1+x)^2e^{-2\tau} + \dots\end{aligned}\quad (10)$$

The terms decaying as $e^{-n\tau}$ in this expansion correspond to quasinormal modes. To see this, let us insert the Ansatz $\phi(\tau, x) = e^{\lambda\tau}\psi(x)$ into equation (4). This yields the quadratic eigenvalue problem

$$(1-x^2)\psi'' - 2x(\lambda+1)\psi' - \lambda(\lambda+1)\psi = 0, \quad (11)$$

whose general solution is

$$\psi(x) = A(1+x)^{-\lambda} + B(1-x)^{-\lambda},$$

where A and B are constants. The quasinormal modes are defined as smooth solutions of equation (11). The requirement of smoothness implies quantization of eigenvalues $\lambda_n = -n$, $n = 0, 1, 2, \dots$. For $\lambda_0 = 0$, the eigenfunction is constant $\psi_0 = 1$, while for $n \geq 1$, there is a two-fold degeneracy

$$\psi_n^{\pm}(x) = (1+x)^n \pm (1-x)^n,$$

where for convenience we took odd and even combinations.

Returning to expansion (10) and defining $\phi_1(0) = \frac{1}{2}(a_0 - b_0)$, $\phi_2(0) = \frac{1}{2}(a_0 + b_0)$, $\phi_1'(0) = a_1 - b_1$, $\phi_2'(0) = a_1 + b_1$, $\phi_1''(0) = 2(a_2 - b_2)$, $\phi_2''(0) = 2(a_2 + b_2)$, \dots , we obtain the late-time behavior as a superposition of quasinormal modes

$$\begin{aligned}\phi(\tau, x) &= a_0 + [a_1\psi_1^+(x) + b_1\psi_1^-(x)]e^{-\tau} \\ &\quad + [a_2\psi_2^+(x) + b_2\psi_2^-(x)]e^{-2\tau} + \dots\end{aligned}$$

It is clear from this expression that no quasinormal modes are excited for the initial data f and g that are supported away from the cosmological horizons $x = \pm 1$.

Remark. It is instructive to compare the above approach with the traditional definition of quasinormal modes as outgoing wave solutions. Using time t and defining the tortoise coordinate $y = \operatorname{arctanh} x$, one transforms equation (3) with $m = 0$ into $\partial_{tt}\phi = \partial_{yy}\phi$. Separating time $\phi(t, y) = e^{\lambda t}\xi(y)$, we get $\xi'' - \lambda^2\xi = 0$, whose general solution is $\xi(y) = Ae^{\lambda y} + Be^{-\lambda y}$. Thus, there are no solutions that are outgoing both at $y = -\infty$ and $y = \infty$. Note that in the coordinates (t, y) , the quasinormal modes $e^{-n\tau}\psi_n^\pm(x)$ are given by $e^{-n(t-y)} \pm e^{-n(t+y)}$.

4. Massive scalar field

Since for $m \neq 0$ there is no d'Alembert formula available, we shall use a different approach, based on the Galerkin method, similar to the one developed by two of us in [10]. This will allow us to solve the initial value problem explicitly for all polynomial initial data.

We begin by expanding the solution in Legendre polynomials $P_n(x)$

$$\phi(\tau, x) = \sum_{n=0}^{\infty} a_n(\tau) P_n(x). \quad (12)$$

Inserting this series into equation (4), we obtain the infinite system of ordinary differential equations for the coefficients $a_n(\tau)$

$$\ddot{a}_n + (2n+1)\dot{a}_n + [n(n+1) + m^2] a_n + 2(2n+1)f_n = 0, \quad (13)$$

where the dot denotes differentiation with respect to τ , and we defined

$$f_n(\tau) \equiv \dot{a}_{n+2}(\tau) + \dot{a}_{n+4}(\tau) + \dots \quad (14)$$

In deriving (13), the projection of the term $2x\partial_{\tau x}\phi$ was obtained using the identity

$$xP'_n(x) = nP_n(x) + \sum_{n-2j \geq 0} (2n+1-4j)P_{n-2j}(x),$$

which follows readily from Bonnet's recursion formula

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

Note that the even and odd modes decouple in system (13).

Assuming that $f_n(\tau)$ is known, one can formally solve equation (13) as an inhomogeneous linear equation with constant coefficients. The characteristic equation for the homogeneous part is

$$r^2 + (2n + 1)r + [n(n + 1) + m^2] = 0,$$

which has roots

$$r = -n - \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4m^2}, \quad (15)$$

hence there are three different cases depending on whether m^2 is less, equal, or larger than $\frac{1}{4}$. To illustrate the method, let us consider the case of $m^2 > \frac{1}{4}$ (the other two cases can be treated in an analogous way). In this case, the solution of the homogeneous part of equation (13) has the form of

$$a_n^{\text{hom}}(\tau) = e^{-\gamma_n \tau} [A_n \cos(\omega \tau) + B_n \sin(\omega \tau)], \quad (16)$$

where $\gamma_n = n + \frac{1}{2}$, $\omega = \frac{1}{2}\sqrt{4m^2 - 1}$ and A_n, B_n are constants. Using the method of variations of constants, one can write the solution of equation (13) for a given f_n as

$$\begin{aligned} a_n(\tau) = a_n^{\text{hom}}(\tau) &+ \frac{4\gamma_n}{\omega} e^{-\gamma_n \tau} \cos(\omega \tau) \int_0^\tau \sin(\omega \tau') e^{\gamma_n \tau'} f_n(\tau') d\tau' \\ &- \frac{4\gamma_n}{\omega} e^{-\gamma_n \tau} \sin(\omega \tau) \int_0^\tau \cos(\omega \tau') e^{\gamma_n \tau'} f_n(\tau') d\tau'. \end{aligned} \quad (17)$$

The constants A_n and B_n can be expressed in terms of the initial values of a_n and \dot{a}_n as follows:

$$A_n = a_n(0), \quad \omega B_n = \dot{a}_n(0) + \gamma_n a_n(0). \quad (18)$$

Suppose now that the initial data are polynomial in x , hence they consist of a finite number of Legendre modes. Let the highest mode number be N . The solutions for a_N and a_{N-1} are then given by the solutions of the homogeneous equation (16). Once these solutions are known, one can easily get solutions for a_{N-2} and a_{N-3} by a straightforward application of formula (17). This procedure can then be iterated backwards, until the solutions for a_0 and a_1 are obtained. For example, for $N = 2$, we have from (16) and (18)

$$\begin{aligned} a_2(\tau) &= e^{-5\tau/2} \left[\frac{5a_2(0) + 2\dot{a}_2(0)}{2\omega} \sin(\omega \tau) + a_2(0) \cos(\omega \tau) \right], \\ a_1(\tau) &= e^{-3\tau/2} \left[\frac{3a_1(0) + 2\dot{a}_1(0)}{2\omega} \sin(\omega \tau) + a_1(0) \cos(\omega \tau) \right], \end{aligned}$$

and, next, inserting $f_0(\tau) = \dot{a}_2(\tau)$ into formula (17), we get

$$e^{-\tau/2} [c_1 \sin(\omega\tau) + c_2 \cos(\omega\tau)] + e^{-5\tau/2} [c_3 \sin(\omega\tau) + c_4 \cos(\omega\tau)] ,$$

where

$$\begin{aligned} c_1 &= \frac{4(\omega^2 + 1)[a_0(0) + 2\dot{a}_0(0)] + (4\omega^2 + 25)a_2(0) - (4\omega^2 - 2)\dot{a}_2(0)}{8\omega(\omega^2 + 1)} , \\ c_2 &= \frac{8(\omega^2 + 1)a_0(0) - (4\omega^2 + 25)a_2(0) - 6\dot{a}_2(0)}{8(\omega^2 + 1)} , \\ c_3 &= \frac{(4\omega^2 + 25)a_2(0) + 2(2\omega^2 + 5)\dot{a}_2(0)}{8\omega(\omega^2 + 1)} , \\ c_4 &= \frac{(4\omega^2 + 25)a_2(0) + 6\dot{a}_2(0)}{8(\omega^2 + 1)} . \end{aligned}$$

The quasinormal modes can be determined as in the massless case by the separation of variables $\phi(\tau, x) = e^{\lambda\tau}\psi(x)$ which, when inserted into (4), leads to the quadratic eigenvalue problem

$$(1 - x^2)\psi'' - 2x(\lambda + 1)\psi' - [\lambda(\lambda + 1) + m^2]\psi = 0 . \quad (19)$$

Searching for solutions in the form of a power series $\psi(x) = \sum_{n=0}^{\infty} b_n x^n$ gives the recurrence relation

$$b_{n+2} = \frac{\lambda^2 + (1 + 2n)\lambda + n(n + 1) + m^2}{(n + 1)(n + 2)} b_n .$$

For the solution to be smooth, it is necessary that the power series terminates at some finite n ; this happens for

$$\lambda = \lambda_n \equiv -\frac{1}{2} \left(2n + 1 \pm \sqrt{1 - 4m^2} \right) , \quad n = 0, 1, \dots \quad (20)$$

The corresponding eigenfunctions $\psi_n(x)$ are then polynomials (but not orthogonal ones). As follows from (20), for $m^2 \leq \frac{1}{2}$ the quasinormal modes are purely damped, while for $m^2 > \frac{1}{4}$, they are damped oscillations.

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